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A COMMENT OF POWER IN n-GROUP

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Abstract. Let $n \geq 2$, let (Q, A) be an $n-$group, $\mathbf{e}$ its $\{1, n\}-$neutral operation /: [6]; 1.3/, and $^{-1}$ its inversing operation /: [7]; 1.3/. Let also Z be an set of all integers. Then, in this paper, we say that a^m $(m \in \in Z)$ is an $m-$th power of the element a in (Q, A) iff: (1) $a^1 \overset{def}{=} a$; (2) $a^{k+1} \overset{def}{=} A(a^k, \overset{n-2}{a}, a), k \geq 1$; (3) $a^{\circ} \overset{def}{=} \mathbf{e}(\overset{n-2}{a})$ and (4) $a^{-k} \overset{def}{=} (\overset{n-2}{a}, a^k)^{-1}$, $k \geq 1$ /:2.3/. Furthermore, for all $a \in Q$ and for all $s \in Z$ the following equality holds: $a^{<s>} = a^{s+1}$, where $a^{<s>}$ well-known is the $s-$th $n-$adic power of the element a in (Q, A) /:2.5/; $< s >= s(n-1)+1$. Among others, in the paper is proved the following proposition. For every $\alpha, \alpha_1, \ldots, \alpha_n \in Z$ $(n \geq 3)$ the following equalities hold:

$$\mathbf{e}(a^{\alpha_1}, \ldots, a^{\alpha_{n-2}}) = a^{-\sum\limits_{i=1}^{n-2} \alpha_i + n - 2},$$

$$(a^{\alpha_1}, \ldots, a^{\alpha_{n-2}}, a^{\alpha})^{-1} = a^{-\alpha - 2\left(\sum\limits_{i=1}^{n-2} \alpha_i - n + 2\right)} \quad \text{and}$$

$$A(a^{\alpha_1}, \ldots, a^{\alpha_n}) = a^{\sum\limits_{i=1}^{n} \alpha_i - n + 2} \quad \text{/:2.7,2.8, footnote 4) /.}$$

1. Preliminaries

1.1. Definition: *Let $n \geq 2$ and let (Q, A) be an $n-$groupoid. We say that (Q, A) is a Dörnte $n-$group [briefly: $n-$group] iff is an $n-$semigroup and an $n-$quasigroup as well.*[1]

1.2. Proposition [10]: *Let $n \geq 2$ and let (Q, A) be an $n-$groupoid. Then the following statements are equivalent: (i) (Q, A) is an $n-$group; (ii) there are mappings $^{-1}$ and $\mathbf{e}$ respectively of the sets Q^{n-1} and Q^{n-2} into the set Q such that the following laws hold in the algebra $(Q, \{A, ^{-1}, \mathbf{e}\})$ [of the type $< n, n-1, n-2 >$]*

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[1] A notion of an $n-$group was introduced by W. Dörnte in [1] as a generalization of the notion of a group.

(a) $A(x_1^{n-2}, A(x_{n-1}^{2n-2}), x_{2n-1}) = A(x_1^{n-1}, A(x_n^{2n-1}))$,
(b) $A(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, x) = x$ *and*
(c) $A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, a) = \mathbf{e}(a_1^{n-2})$; *and*

(iii) there are mappings $^{-1}$ *and* $\mathbf{e}$ *respectively of the sets* Q^{n-1} *and* Q^{n-2} *into the set* Q *such that the following laws hold in the algebra* $(Q, \{A, ^{-1}, \mathbf{e}\})$ *[of the type* $< n, n-1, n-2 >$*]*

($\bar{a}$) $A(A(x_1^n), x_{n+1}^{2n-1}) = A(x_1, A(x_2^{n+1}), x_{n+2}^{2n-1})$,
($\bar{b}$) $A(x, a_1^{n-2}, \mathbf{e}(a_1^{n-2})) = x$ *and*
($\bar{c}$) $A(a, a_1^{n-2}, (a_1^{n-2}, a)^{-1}) = \mathbf{e}(a_1^{n-2})$.

1.3. Remarks: $\mathbf{e}$ is an $\{1, n\}$−neutral operation of n−grupoid (Q, A) iff algebra $(Q, \{A, \mathbf{e}\})$ of type $< n, n-2 >$ satisfies the laws (b) and $(\bar{b})$ from 1.2 *[*:[6]*]*. The notion of $\{i, j\}$−neutral operation $(i, j \in \{1, \dots, n\}, i < j)$ of an n−groupoid is defined in a similar way *[*:[6]*]*. Every n−groupoid there is **at most one** $\{i, j\}$−neutral operations *[*:[6]*]*. In every n−group, $n \geq 2$, there is a $\{1, n\}$−neutral operation *[*:[6]*]*. There are n−groups without $\{i, j\}$−neutral operations with $\{i, j\} \neq \{1, n\}$*[*:[8]*]*. In [8], n−groups with $\{i, j\}$−neutral operations, for $\{i, j\} \neq \{1, n\}$ are described. Operation $^{-1}$ from 1.2 *[*$(c), (\bar{c})$*]* is a generalization of the inversing operation in a group. In fact, if (Q, A) is an n−group, $n \geq 2$, then for every $a \in Q$ and for every sequence a_1^{n-2} over Q is

$$\left(a_1^{n-2}, a\right)^{-1} \stackrel{def}{=} \mathsf{E}\left(a_1^{n-2}, a, a_1^{n-2}\right),$$

where E is an $\{1, 2n-1\}$−neutral operation of the $(2n-1)$−group $(Q, \overset{2}{A})$; $\overset{2}{A}(x_1^{2n-1}) \stackrel{def}{=} A(A(x_1^n), x_{n+1}^{2n-1})$ *[*:[7]*]*. (For $n = 2$, $a^{-1} = \mathsf{E}(a)$; a^{-1} is the inverse element of the element a with respect to the neutral element $\mathbf{e}(\emptyset)$ of the group (Q, A).)

1.4. Proposition *(Hosszú-Gluskin Theorem) [***2-3***]: For every* n*−group* (Q, A)*,* $n \geq 3$*, there is an algebra* $(Q, \{\cdot, \varphi, b\})$ *such that the following statements hold:* 1° $(Q, \cdot)$ *is a group;* 2° $\varphi \in Aut(Q, \cdot)$; 3° $\varphi(b) = b$; 4° *for every* $x \in Q$, $\varphi^{n-1}(x) \cdot b = b \cdot x$; *and* 5° *for every* $x_1^n \in Q$, $A(x_1^n) = x_1 \cdot \varphi(x_2) \cdot \dots \cdot \varphi^{n-1}(x_n) \cdot b$.

1.5. Definition [**9**]: *We say that an algebra* $(Q, \{\cdot, \varphi, b\})$ *is a Hosszú-Gluskin algebra of order* $n (n \geq 3)$ *[briefly: nHG-algebra] iff* 1° − 4° *from 1.4 hold. In addition, we say that an nHG- algebra* $(Q, \{\cdot, \varphi, b\}))$ *is* **associated** *to the* n*−group* (Q, A) *iff* 5° *from 1.4 holds.*

1.6. Proposition [9]: *Let $n \geq 3$, let (Q, A) be an $n-$group, and* $\mathbf{e}$ *its* $\{1,n\}-$ *neutral operation. Further on, let c_1^{n-2} be an arbitrary sequence over Q and let for every $x, y \in Q$*

$$B_{(c_1^{n-2})}(x,y) \stackrel{def}{=} A\left(x, c_1^{n-2}, y\right),$$

$$\varphi_{(c_1^{n-2})}(x) \stackrel{def}{=} A\left(\mathbf{e}(c_1^{n-2}), x, c^{n-2}\right) \text{ and}$$

$$b_{(c_1^{n-2})} \stackrel{def}{=} A\left(\overline{\overset{n}{\mathbf{e}(c_1^{n-2})|}}\right).$$

Then, the following statements hold

(*i*) $(Q, \{B_{(c_1^{n-2})}, \varphi_{(c_1^{n-2})}, b_{(c_1^{n-2})})\}$ *is an $nHG-$algebra associated to the $n-$group (Q, A); and*

(*ii*) $C_A \stackrel{def}{=} \{(Q, \{B_{(c_1^{n-2})}, \varphi_{(c_1^{n-2})}, b_{(c_1^{n-2})}\}) | c_1^{n-2}$ *is a* **sequence** *over $Q\}$ is the set of all $nHG-$algebras* **associated** *to the $n-$group (Q, A).*

1.7. Definition: *Let (Q, B) be an $n-$groupoid and $n \geq 2$. Then 1) $\overset{1}{B} \stackrel{def}{=} B$; and 2) for every $k \in N$ and for every $x_1^{(k+1)(n-1)+1} \in Q$*

$$\overset{k+1}{B}\left(x_1^{(k+1)(n-1)+1}\right) \stackrel{def}{=} B\left(\overset{k}{B}\left((x_1^{k(n-1)+1}\right), x_{k(n-1)+2}^{(k+1)(n-1)+1}\right).$$

1.8. Proposition: *Let (Q, B) be an $n-$semigroup, $n \geq 2$ and $(i, j) \in \in N^2$. Then, for every $x_1^{k(n-1)+1} \in Q$ and for every $t \in \{1, \ldots, i(n-1)+1\}$ the following equality holds*

$$\overset{i+j}{B}\left(x_1^{(i+j)(n-1)+1}\right) = \overset{i}{B}\left(x_1^{t-1}, \overset{j}{B}\left(x_t^{t+j(n-1)}\right), x_{t+j(n-1)+1}^{(i+j)(n-1)+1}\right).$$

2. Results

2.1. Definition :[2] *Let $n \geq 2$ and let (Q, A) be an $n-$group. Let, also, Z be an set of all integers. Then we say that $a^{<s>} (s \in Z)$ is the $s-$th $n-$adic power of the element a in (Q, A) iff:*

(*a*) $a^{<s>} \stackrel{def}{=} a, \ s = 0;$

(*b*) $a^{<s>} \stackrel{def}{=} \overset{s}{A}(\overset{s(n-1)+1}{a}), \ s > 0;$ and

(*c*) $a^{<s>} \stackrel{def}{=} x, s < 0,$ where $\overset{-s}{A}(x, \overset{-s(n-1)}{a}) = a$ [3]

[2]Rusakov S. A., 1978. Information from [5].

[3]In [5] S. A. Rusakov uses $\left(\overset{k(n-1)+1}{a}\right)$ instead of $\overset{k}{A}\left(\overset{k(n-1)+1}{a}\right), k > 0.$

2.2. Remark: For $s > 0$ the following equality holds: $< s >= s(n-1)+1$ [4]. Moreover, s is the number of appearances of the operation A in the description of the power $a^{<s>}$ *[:(b)]* (for all $n \geq 2$).

2.3. Definition: *Let $n \geq 2$. Let also (Q, A) be an $n-$group, $\mathbf{e}$ its $\{1,n\}-$neutral operation [:1.3] and $^{-1}$ its inversing operation [:1.3]. Then we shall say that $a^m (m \in Z)$ is the $m-$th power of the element a in (Q, A) iff:*

(1) $a^1 \overset{def}{=} a$;

(2) $a^{k+1} \overset{def}{=} A(a^k, \overset{n-2}{a}, a),\ k \geq 1$;

(3) $a^\circ \overset{def}{=} \mathbf{e}(\overset{n-2}{a})$ and

(4) $a^{-k} \overset{def}{=} (\overset{n-2}{a}, a^k)^{-1}, k \geq 1$.

2.4. Remark: For $n = 2$, the conditions (1)–(4) reduce to the conditions:

($\hat{1}$) $a^1 \overset{def}{=} a$;

($\hat{2}$) $a^{k+1} \overset{def}{=} A(a^k, a),\ k \geq 1$;

($\hat{3}$) $a^\circ \overset{def}{=} e[= \mathbf{e}(\emptyset);\ \overset{n-2}{a} = \overset{0}{a} = \emptyset\ ]$ and

($\hat{4}$) $a^{-k} \overset{def}{=} (a^k)^{-1}, k \geq 1$.

2.5 Proposition: *Let $n \geq 2$ and let (Q, A) be an $n-$group. Let, also, Z be an set of all integers. Then for all $a \in Q$ and for all $s \in Z$ the following equality holds*

$$a^{<s>} = a^{s+1}$$

[:2.1,2.3].

Sketch of the proof.

$s = 0:$ *a)* $a^1 = a^{<0>}$ *[:2.3 – (1), 2.1 – (a)].*

$s > 0:$ *b)* $a^m = \overset{m-1}{A}(\overset{(m-1)(n-1)+1}{a}),\ m \geq 2$ *[:2.3 – (1), (2); 1.7, 1.8].*

c) $a^{<s>} = \overset{s}{A}(\overset{s(n-1)+1}{a})$ *[:2.1 – (b)].*

d) $a^{s+1} = \overset{s}{A}(\overset{s(n-1)+1}{a}) = a^{<s>}$ *[:b), c)].*

$s < 0:$ *e)* $s = -1:$

$A(a^{<-1>}, \overset{n-1}{a}) = a \Leftrightarrow A(a^{<-1>}, \overset{n-2}{a}, a) = a,$

$a^{<-1>} = \mathbf{e}(\overset{n-2}{a}) \Leftrightarrow a^{<-1>} = a^0$ *[:2.1 – (c), 1.3, 3.3 – (3)].*

f) $s = -k,\ k = 2:$

$\overset{2}{A}(a^{<-2>}, \overset{n-2}{a}, a, \overset{n-2}{a}, a) = a \Leftrightarrow A(a^{<-2>}, \overset{n-2}{a}, A(a^1, \overset{n-2}{a}, a)) = a,$

$(\overset{n-2}{a}, a^1)^{-1} = a^{<-2>} \Leftrightarrow a^{-1} = a^{<-2>}$ *[:2.1 – (c), 1.7, 1.8, 1.3, 2.3 – (1), 2.3 – (4)].*

$g)\ s = -k,\ k > 2:$

$\overset{k}{A}(a^{<-k>}, \overset{k(n-1)}{a}) = a \Leftrightarrow \overset{k}{A}(a^{<-k>}, \overset{n-2}{a}, \overset{(k-2)(n-1)+1}{a}, \overset{n-2}{a}, a) = a \Leftrightarrow$

$\overset{2}{A}(a^{<-k>}, \overset{n-2}{a}, \overset{k-2}{A}(\overset{(k-2)(n-1)+1}{a}), \overset{n-2}{a}, a) = a \Leftrightarrow$

$\overset{2}{A}(a^{<-k>}, \overset{n-2}{a}, a^{k-1}, \overset{n-2}{a}, a) = a \Leftrightarrow$

$A(a^{<-k>}, \overset{n-2}{a}, A(a^{k-1}, \overset{n-2}{a}, a)) = a,$

$(\overset{n-2}{a}, a^{k-1})^{-1} = a^{<-k>} \Leftrightarrow a^{-k+1} = a^{<-k>}$ [:2.1-(c),1.7,1.8,b),1.3,2.3-(4)]. □

Let $n \geq 3$, (Q, A) be an n-group, $^{-1}$ its inversing operation [:1.3] and **e** its $\{1, n\}$-neutral operation [:1.3]. Let also a be an arbitrary element of the set Q and for all $x, y \in Q$ let:

(5) $x \square y \overset{def}{=} A(x, \overset{n-2}{a}, y)$,

(6) $x^{-1} \overset{def}{=} (\overset{n-2}{a}, x)^{-1}$ and

(7) $e_\square \overset{def}{=} \mathbf{e}(\overset{n-2}{a})$.

Then, $(Q, \square)$ is a group with the inversing operation $^{-1}$ and the neutral element $e_\square$ [:1.2,1.3]. By the convention with (5)–(7), the conditions (1)–(4) can be formulated in the following way:

$(\bar{1})$ $a^1 \overset{def}{=} a$;

$(\bar{2})$ $a^{k+1} \overset{def}{=} a^k \square a,\ k \geq 1$;

$(\bar{3})$ $a^\circ \overset{def}{=} e_\square$ and

$(\bar{4})$ $a^{-k} \overset{def}{=} (a^k)^{-1},\ k \geq 1$.

Hence, the following proposition is fulfilled:

2.6. Theorem: *Let $n \geq 3$, $(Q, \{A, ^{-1}, \mathbf{e}\})$ be an n-group as variety of type $< n, n-1, n-2 >$ [:1.2,1.3], a be an arbitrary element from Q and $(Q, \{\square, ^{-1}, e_\square\})$ the group defined by (5)-(7). Let, also, Z be an set of all integers. Then: $a^m (m \in Z)$ is the m-th power of the element a in the n-group $(Q, \{A, ^{-1}, \mathbf{e}\})$ iff a^m is the m-th power of a in the group $(Q, \{\square, ^{-1}, e_\square\})$.* □

2.7. Theorem: *Let $n \geq 3$, $(Q, \{A, ^{-1}, \mathbf{e}\})$ be an n-group as variety of type $< n, n-1, n-2 >$ [:1.2,1.3], a be an arbitrary element from Q. Let, also, Z be an set of all integers. Then for every $\alpha, \alpha_1, \ldots, \alpha_n \in Z$ the following equalities hold*

$$(8) \qquad A(a^{\alpha_1}, \ldots, a^{\alpha_n}) = a^{\sum_{i=1}^{n} \alpha_i - n + 2}\ [4],$$

$$(9) \qquad (a^{\alpha_1}, \ldots, a^{\alpha_{n-2}}, a^{\alpha})^{-1} = a^{-\alpha - 2(\sum_{i=1}^{n-2} \alpha_i - n + 2)}$$

[4] $A(a^{<\alpha_1>}, \ldots, a^{<\alpha_n>}) = a^{<\alpha_1 + \cdots + \alpha_n + 1>}$; for $\alpha_1, \ldots, \alpha_n \in N \cup \{0\}$ see, e.g., [4].

$$(10) \qquad \mathbf{e}(a^{\alpha_1}, \ldots, a^{\alpha_{n-2}}) = a^{-\sum_{i=1}^{n-2}\alpha_i + n-2}.$$

Proof. 1) Let $n \geq 3$, (Q, A) be an $n-$group, $^{-1}$ its inversing operation / 1.3/ and $\mathbf{e}$ its $\{1, n\}-$neutral operation / 1.3/. Let, also, $(Q, \{\cdot, \varphi, b\})$ be an arbitrary $nHG-$ algebra associated to the $n-$group (Q, A) / 1.5/. Further on, let $^{-1}$ be an inversing operation of the group $(Q, \cdot)$. Then, by 1.2,1.3 an 1.5, we conclude that for all $c \in Q$ and for everysequence c_1^{n-2} over Q the following equalities hold

$$\mathbf{e}(c_1^{n-2}) = (\varphi(c_1) \cdot \dots \cdot \varphi^{n-2}(c_{n-2}) \cdot b)^{-1} \text{ and}$$
$$(c_1^{n-2}, c) = (\varphi(c_1) \cdot \dots \cdot \varphi^{n-2}(c_{n-2}) \cdot b \cdot c \cdot \varphi(c_1) \cdot \dots \cdot \varphi^{n-2}(c_{n-2}) \cdot b)^{-1}.$$

2) Let a be an arbitrary element from Q and for every $x, y \in Q$ let

$$x \Box y \stackrel{def}{=} A(x, \overset{n-2}{a}, y) \ [(5)],$$
$$\varphi_\Box(x) \stackrel{def}{=} A(\mathbf{e}(\overset{n-2}{a}), x, \overset{n-2}{a}) \text{ and}$$
$$b_\Box \stackrel{def}{=} A\left(\overline{\mathbf{e}(\overset{n-2}{a})}^{\,n}\right).$$

Then $(Q, \{\Box, \varphi_\Box, b_\Box\})$ is an $nHG-$ algebra associated to the $n-$group (Q, A) /:1.6/. Moreover, for every $m \in Z$ the equality

$$\varphi_\Box(a^m) = a^m$$

holds.

Indeed:

2_1) Let $m = 1$. Then the following sequence of equalities holds

$$\begin{aligned}\varphi_\Box(a^1) &= \varphi_\Box(a) = A(\mathbf{e}(\overset{n-2}{a}), a, \overset{n-2}{a}) \\ &= A(\mathbf{e}(\overset{n-2}{a}), \overset{n-2}{a}, a) = a = a^1\end{aligned}$$

/:(1),1.2,1.3/.

2_2) Let $m = k \geq 2$. Then the following sequence of equalities holds

$$\begin{aligned}\varphi_\Box(a^k) &= A(\mathbf{e}(\overset{n-2}{a}), a^k, \overset{n-2}{a}) \\ &= A(\mathbf{e}(\overset{n-2}{a}), \overset{k-1}{A}(\overset{(k-1)(n-1)+1}{a}), \overset{n-2}{a}) \\ &= A(\mathbf{e}(\overset{n-2}{a}), \overset{n-2}{a}, \overset{k-1}{A}(\overset{(k-1)(n-1)+1}{a})), \\ &= \overset{k-1}{A}(\overset{(k-1)(n-1)+1}{a}) = a^k\end{aligned}$$

/:2.3-(1),(2);1.7,1.8,1.3/.

2_3) Let $m = 0$. Then the following sequence of equalities holds

$$\begin{aligned}\varphi_\Box(a^\circ) &= \varphi_\Box(\mathbf{e}(\overset{n-2}{a})) = A(\mathbf{e}(\overset{n-2}{a}), \mathbf{e}(\overset{n-2}{a}), \overset{n-2}{a}) \\ &= \mathbf{e}(\overset{n-2}{a}) = a^\circ\end{aligned}$$

/:(3); $F(x, c_1^{n-2}) = A(x, \mathbf{e}(c_1^{n-2}), c_1^{n-2}) \Rightarrow A(F(x, c_1^{n-2}), \mathbf{e}(c_1^{n-2}), c_1^{n-2}) =$

$A(A(x, \mathbf{e}(c_1^{n-2}), c_1^{n-2}), \mathbf{e}(c_1^{n-2}), c_1^{n-2}) \Rightarrow A(F(x, c_1^{n-2}), \mathbf{e}(c_1^{n-2}), c_1^{n-2}) =$
$A(x, \mathbf{e}(c_1^{n-2}), c_1^{n-2}) \Rightarrow F(x, c_1^{n-2}) = x$, 1.2,1.3/.

2_4) Let $m = -1$. Then the following sequence of equalities holds

$$\begin{aligned} \varphi_{\Box}(a^{-1}) &= \varphi_{\Box}((\overset{n-2}{a}a)^{-1}) = A(\mathbf{e}(\overset{n-2}{a}), (\overset{n-1}{a})^{-1}, \overset{n-2}{a}) \\ &= A(A((\overset{n-1}{a})^{-1}, \overset{n-2}{a}, a), (\overset{n-1}{a})^{-1}, \overset{n-2}{a}) \\ &= A((\overset{n-1}{a})^{-1}, A(a, \overset{n-2}{a}, (\overset{n-1}{a})^{-1}), \overset{n-2}{a}) \\ &= A((\overset{n-1}{a})^{-1}, \mathbf{e}(\overset{n-2}{a}), \overset{n-2}{a}) \\ &= (\overset{n-1}{a})^{-1} = (\overset{n-2}{a}, a^1)^{-1} = a^{-1} \end{aligned}$$

/:(4);1.3; $A(x, \mathbf{e}(c_1^{n-2}), c_1^{n-2}) = x$, 2_3)/.

2_5) Let $m = -k$ and $k \geq 2$. Then the following sequence of equalities holds

$$\begin{aligned} \varphi_{\Box}(a^{-k}) &= \varphi_{\Box}((\overset{n-2}{a}, a^k)^{-1}) = A(\mathbf{e}(\overset{n-2}{a}), (\overset{n-2}{a}, a^k)^{-1}, \overset{n-2}{a}) \\ &= A(A((\overset{n-2}{a}, a^k)^{-1}, \overset{n-2}{a}, a^k), (\overset{n-2}{a}, a^k)^{-1}, \overset{n-2}{a}) \\ &= A((\overset{n-2}{a}, a^k)^{-1}, A(\overset{n-2}{a}, a^k, (\overset{n-2}{a}, a^k)^{-1}), \overset{n-2}{a}) \\ &= A((\overset{n-2}{a}, a^k)^{-1}, A(\overset{n-2}{a}, \overset{k-1}{A}(\overset{(k-1)(n-1)+1}{a}), (\overset{n-2}{a}, a^k)^{-1}), \overset{n-2}{a}) \\ &= A((\overset{n-2}{a}, a^k)^{-1}, A(\overset{k-1}{A}(\overset{(k-1)(n-1)+1}{a}), \overset{n-2}{a}, (\overset{n-2}{a}, a^k)^{-1}), \overset{n-2}{a}) \\ &= A((\overset{n-2}{a}, a^k)^{-1}, A(a^k, \overset{n-2}{a}, (\overset{n-2}{a}, a^k)^{-1}), \overset{n-2}{a}) \\ &= A((\overset{n-2}{a}, a^k)^{-1}, \mathbf{e}(\overset{n-2}{a}), \overset{n-2}{a}) \\ &= (\overset{n-2}{a}, a^k)^{-1} = a^{-k} \end{aligned}$$

/:1.2,1.3,(4),1.7,1.8, $A(x, \mathbf{e}(c_1^{n-2}), c_1^{n-2}) = x - 2_3$)/.

3) By 2), 1.5 and 2.6, we conclude that the following sequence of equalities holds

$$\begin{aligned} a\Box a &= A(a, \overset{n-2}{a}, a) = A(\overline{a^1|}^{\,n}) \\ &= a^1\Box \ldots \Box a^1 \Box b_{\Box} \\ &= a\Box \ldots \Box a \Box b_{\Box}, \end{aligned}$$

and hence we conclude that

$$b_{\Box} = a^{-(n-2)}.$$

4) Finally, by proposition from 1)–3), Theorem 2.6 and 1.5, we conclude

that for every $\alpha, \alpha_1, \dots, \alpha_n \in Z$ the following equalities hold

$$A(a^{\alpha_1}, \dots, a^{\alpha_n}) = a^{\alpha_1} \square \dots \square a^{\alpha_n} \square a^{-(n-2)} = a^{-\sum_{i=1}^{n-2} \alpha_i - (n-2)},$$

$$(a^{\alpha_1}, \dots, a^{\alpha_{n-2}, a^{\alpha}})^{-1} = (a^{\alpha_1} \square \dots \square a^{\alpha_{n-2}} \square a^{-(n-2)} \square 2a^{\alpha} \square a^{\alpha_1} \square \dots$$

$$\dots \square a^{\alpha_{n-2}} \square a^{-(n-2)})^{-1} = a^{-\alpha - 2(\sum_{i=1}^{n-2} \alpha_i - (n-2))} \quad \text{and}$$

$$\mathbf{e}(a^{\alpha_1}, \dots, a^{\alpha_{n-2}}) = (a^{\alpha_1} \square \dots \square a^{\alpha_{n-2}} \square a^{-(n-2)})^{-1} = a^{-\sum_{i=1}^{n-2} \alpha_i + n - 2}.$$

□

2.8. Remark: For $n = 2$, the equality (8) reduce to the well-known equality

$$A(a^{\alpha_1}, a^{\alpha_2}) = a^{\alpha_1 + \alpha_2}.$$

Moreover, for $n = 2$, by convection

$$\sum_{i=1}^{0} \stackrel{def}{=} 0,$$

the equalities (9) and (10) reduce to the well-known equalities

$$(a^{\alpha})^{-1} = a^{-\alpha} \quad \text{and}$$
$$\mathbf{e}(\emptyset) = a^{\circ},$$

where $\mathbf{e}(\emptyset)$ is a neutral element of the group (Q, A). □

2.9. Example: Let $(\{1,2,3,4\}, \cdot)$ be the Klein's group defined by the table

$\cdot$	1	2	3	4
1	1	2	3	4
2	2	1	4	3
3	3	4	1	2
4	4	3	2	1

and $^{-1}$ its inversing operation. Let also the permutation φ be defined by the table

φ	1	2	3	4
	1	2	4	3

;

$\varphi \in Aut(\{1,2,3,4\}, \cdot)$ $\varphi(2) = 2$, $\varphi^2 = \{(x,x) | x \in \{1,2,3,4\}$. Then, $(\{1,2,3,4\}, A)$, where

$$A(x,y,z) \stackrel{def}{=} x \cdot \varphi(y) \cdot z \cdot 2$$

for every $x, y, z \in \{1, 2, 3, 4\}$, is a 3-group, and for every $a, c \in \{1, 2, 3, 4\}$ the following equalities hold

$$\mathbf{e}(c) = (\varphi(c) \cdot 2)^{-1} \quad \text{and}$$
$$(c, a)^{-1} = [(\varphi(c) \cdot 2 \cdot a \cdot \varphi(c) \cdot 2)^{-1} = a^{-1} =]a.$$

In addition, the following series of equalities holds
$x \cdot_1 y = A(x, 1, y) = x \cdot \varphi(1) \cdot y \cdot 2 = x \cdot 2 \cdot y,$
$x \cdot_2 y = A(x, 2, y) = x \cdot y,$
$x \cdot_3 y = A(x, 3, y) = x \cdot \varphi(3) \cdot y \cdot 2 = x \cdot 3 \cdot y$ and
$x \cdot_4 y = A(x, 4, y) = x \cdot 4 \cdot y.$

$$[\varphi_1 = \varphi_2 = \varphi,\ \varphi_3 = \varphi_4 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix},\ b_1 = 1,\ b_2 = 2,\ b_3 = 3,\ b_4 = 4.]$$

Finally, by Theorem 2.6, we conclude that the following sequence of equalities holds
$1^1 = 1,\ 1^2 = 1 \cdot_1 1 = 2,\ 1^0 = [\mathbf{e}(1) =]\,2,\ 1^{-1} = 1,\ 2^{-1} = 2;$
$2^1 = 2,\ 2^2 = 2 \cdot_2 2 = 1,\ 2^0 = 1,\ 1^{-1} = 1,\ 2^{-1} = 2;$
$3^1 = 3,\ 3^2 = 3 \cdot_3 3 = 3,\ 3^0 = 3,\ 3^{-1} = 3$ and
$4^1 = 4,\ 4^2 = 4 \cdot_4 4 = 4,\ 4^0 = 4,\ 4^{-1} = 4.$ □

2.10. Remark: Power and order of elements in n-group have been also described in the following papers [11-15]. W. A. Dudek has pointed my attention to this fact.

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